1. Recall the population model with carrying capacity, $\frac{d}{dt}P(t) = rP(t)(K - P(t))$. The solution to this problem is given by the function

$$P(t) = \frac{K}{1 + \left(\frac{K}{P(0)} - 1\right) e^{-Krt}}.$$

   (a) Taking $K = 50, r = 1/50$, run the code with $P(0) = 100$ and $P(0) = 10$ and explain why these graphs are consistent with our intuition.

   (b) Using the equation of $P(t)$ provided above, using your algebraic skills, verify that if $P(0) = K$ then $P(t) = K$, and if $P(0) = 0$ then $P(t) = 0$. (You may need to rearrange the equation!)

2. (a) (**Allee Effect**) So far our models have been unable describe population decline of endangered species. We now develop a model with this additional property. We seek a function $f(P(t))$, where $\frac{d}{dt}P(t) = f(P(t))$, such that

$$\begin{cases} f(0) = 0 \quad f(A) = 0 \quad f(K) = 0, \\ f(P(t)) < 0 \quad \text{for } 0 < P(t) < A < K, \\ f(P(t)) > 0 \quad \text{for } A < P(t) < K, \\ f(P(t)) < 0 \quad \text{for } P(t) > K. \end{cases}$$

   Hint: Start with $f(P(t)) = aP(t)^3 + bP(t)^2 + cP(t) + d$ *and then solve for* a, b, c, d.

   (b) Once you have the model, verify that it predicts the correct behaviour using the Allee effect code provided. Take K=50, A=10 and r=0.001, then choose $P(0) \geq 0$ at different values to see if it is consistent with the properties described above!

3. (Bonus Problem) In this problem we study a new model which includes harvesting. Let $P(t)$ denote the population of fish off the coast of Vancouver Island. Let H denote the maximum fishing rate of the entire fleet of ships, and let A denote the amount of fatigue the fisherman have, meaning if A=0 the fishermen are fishing quickly since they are full of energy and if A is large they are fishing slowly due to fatigue. The change of fish population is given by the model

$$\frac{d}{dt}P(t) = P(t)(1 - P(t)) - \frac{HP(t)}{A + P(t)}, \tag{1}$$

   where $H \geq 0$ and $A \geq 0$.

   (a) Describe the fishing model when $H = 0$.

   (b) The harvesting rate in this model is given by $\frac{HP(t)}{A + P(t)}$, describe what happens when $A = 0$?

   (c) Find all fixed points of this model.

(d) Show that if $H > \frac{(A+1)^2}{4}$, then the model only has one fixed point.

(e) Using the code provided, can you find choices of A and H which lead to overfishing? Sustainable fishing (the fish population remains positive)? Can you find choices of A and H leading to multiple positive fixed points? If so, describe what happens to the fish population for different $P(0)$.