# Project on Black Holes

The project will consist of one of: parts 1) and 3); parts 2) and 3; parts 1), 2) and 3).

1) (long) Use energy to derive the coordinate speed

$$\frac{\Delta r}{\Delta t} = -c\left(1 - \frac{r_s}{r}\right)\sqrt{\frac{r_s}{r}}$$

for something that falls towards a black hole.

2) (medium) Use expressions given in the black hole lecture and

$$\frac{\Delta r}{\Delta t} = -c\left(1 - \frac{r_s}{r}\right)\sqrt{\frac{r_s}{r}}$$

to show that when $v = c$ at $r = r_s$ for something that falls towards a black hole. This means that the size of a black hole is given by the Schwarzschild radius

$$r_s = \frac{2GM}{c^2}.$$

3) (short) Give a physical interpretation of $\Delta r/\Delta t$ for something that falls towards a black hole. (short)

**Strategy for preparation time**. First decide (as a group) which of 1) and 2) that you definitely want to do. Do this part first. Next do 3). If you think that you have time, do the part that you omitted.

**Strategy for presentation**
Decide how many of the mathematical steps that you want to show. You might want to show all of them, or you might only want to show some of them.

If you do both of parts 1) and 2), decide which you want be first in your presentation. If you present 1) first, you might want to say something like "this will be used later when we find the size of a black hole." If you present 2) first, you might want to say something like "We will talk about $\Delta r/\Delta t$ later."

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| $\Delta h = \frac{\Delta r}{\sqrt{1-\frac{r_S}{r}}}$ | Hoover's inward distance | $c = 2.998 \times 10^8\,\mathrm{m\,/\,s}$ | speed of light |
| $\Delta T_H = \Delta t\sqrt{1-\frac{r_S}{r}}$ | Hoover's time | $G = 6.67 \times 10^{-11}$ | gravitational constant |
| $r_s = \frac{2GM}{c^2}$ | Schwarzschild radius | $M_S = 1.989 \times 10^{30}\,\mathrm{kg}$ | Sun's mass |

**Part 1)** Use energy to derive expression,

$$\frac{\Delta r}{\Delta t} = -c\left(1-\frac{r_s}{r}\right)\sqrt{\frac{r_s}{r}},$$

for the coordinate speed of a dropped ball was use. **This project** uses energy to derive this expression.

There are 3 objects in this part: 1) Farzana, who hovers at a distance far from the black hole; 2) a mass $m$ ball that Farzana lets fall from rest towards the mass $M$ black hole, 3) Hoover, who hovers above the black hole at coordinate position $r$.

According to Farzana, the ball's energy $E$ is constant throughout its fall, and is given by

$$E = mc^2\left(1-\frac{r_s}{r}\right)\frac{\Delta t}{\Delta T}, \tag{1}$$

where $\Delta T$ is measured by a watch strapped to the ball.

Initially the ball is far from the black hole, so $r \gg r_s$, and $r_s/r \approx 0$. Also, far from the black hole, coordinate time $t$ and watch time $T$ are the same.

**According** to Farzana, what is the energy of the ball at the instant that she drops it? Do you recognize this very famous equation?.

As the ball falls, its energy $E = mc^2$, according to Farzana, does not change. As the falls, its $r$ position changes, and it is no longer true that $r_s/r \approx 0$ and $\Delta t \approx \Delta T$.

**Use** $E = mc^2$ in (1), and solve for (isolate) $\Delta T$.

$$mc^2 = mc^2\left(1-\frac{r_s}{r}\right)\frac{\Delta t}{\Delta T}.$$

**Use** your expression for $\Delta T$ and $\Delta s = c\Delta T$ in the spacetime interval expression.

$$(\Delta s)^2 = c^2\left(1-\frac{r_S}{r}\right)(\Delta t)^2 - \frac{(\Delta r)^2}{\left(1-\frac{r_S}{r}\right)}.$$

**Expand** $\left(1-\frac{r_s}{r}\right)^2$ in the expression.

**Take** all the terms involving $(\Delta t)^2$ to one side.

**Factor** out the $(\Delta t)^2$, and simplify,

**Factor** out $\frac{r_s}{r}$ from the term multiplying $(\Delta t)^2$.

**Solve** for $\left(\frac{\Delta r}{\Delta t}\right)^2$.

**Take** the square root of both sides of the equation. Should this be negative or positive?

**Part 2)** Find where the escape speed equals the speed of light $c$. This is the event horizon of the black hole.

A ball that is at a great distance from the black hole falls from rest towards the black hole;

2) Hoover, who hovers above the black hole at coordinate position, and who sees the ball fall past him (diagram?)

When the ball is near Hoover, it changes height by $\Delta h$ during a time $\Delta T_H$ on Hoover's watch. The speed with which the ball whizzes by Hoover is

$$v = \frac{\Delta h}{\Delta T_H} \tag{2}$$

a) Substitute the expressions for $\Delta h$ and $\Delta T_H$ given in the lecture (and above), into (2). Simplify the expression for $v$ as much as possible.

b) An expression for the coordinate speed of the ball, $\Delta r/\Delta t$ is

$$\frac{\Delta r}{\Delta t} = -c\left(1 - \frac{r_s}{r}\right)\sqrt{\frac{r_s}{r}}. \tag{3}$$

Substitute (3) into the expression for $v$ in part a), and simplify.

c) To find the event horizon, suppose Hoover hovers near the event horizon, so that $v = c$. Use this is in your equation for the previous part, and solve for $r$. Hint: square both sides of the resulting equation.

**Part 3)**. What happens to

$$\frac{\Delta r}{\Delta t} = -c\left(1 - \frac{r_s}{r}\right)\sqrt{\frac{r_s}{r}},$$

as $r$ approaches the Schwarzschild radius $r_s$?

Since $\Delta t$ is time according to Farzana, what will Farzana see as she watches the ball?